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On the Universality of Geometrical and Transport Exponents of Rigidity Percolation

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We develop a three-parameter position-space renormalization group method and investigate the universality of geometrical and transport exponents of rigidity (vector) percolation in two dimensions. To do this, we study site-bond percolation in which sites and bonds are randomly and independently occupied with probabilities s and b, respectively. The global flow diagram of the renormalization transformation is obtained which shows that the *geometrical exponents* of the rigid clusters in both site and bond percolation belong to the same universality class, and possibly that of random (scalar) percolation. However, if we use the same renormalization transformation to calculate the critical exponents of the elastic moduli of the system in bond and site percolation, we find them to be very different (although the corresponding values of the correlation length exponent are the same). This indicates that the critical exponent of the elastic moduli of rigidity percolation may not be universal, which is consistent with some of the recent numerical simulations.

KEY WORDS: Rigidity percolation; elasticity; scalar percolation; universality.

1. INTRODUCTION

For the past two decades random percolation networks$^{(1)}$ have been an important tool for the investigation of transport processes in disordered systems, such as porous media,$^{(2)}$ gel polymers,$^{(3,4)}$ and composite solids.$^{(5)}$ Two of the most commonly studied percolation processes are bond and site

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percolation. In bond percolation (BP) one removes at random a fraction of the bonds, whereas in site percolation (SP) sites (and all bonds connected to them) are removed randomly. Much of the attention has been focused on the properties of percolation networks near the percolation threshold p_c of the network. Consider a percolation network in which a fraction p of bonds (or sites) are present and the rest are absent. Near p_c the correlation length ξ_p diverges as

$$\xi_p \sim (p - p_c)^{-\nu} \tag{1}$$

Scalar transport properties of percolation networks can be defined in a straightforward way. If we assign a finite conductance to the present bonds and a zero conductance (infinite resistance) to the absent bonds, then, near p_c the bulk conductivity σ of the network vanishes as

$$\sigma \sim (p - p_c)^t \tag{2}$$

The backbone of the network, i.e., the current-carrying part of the sample-spanning percolation cluster, has a fractal structure for any length scale less than ξ_p with a fractal dimension D_B. The fractal dimension D_B and the exponent ν are completely universal and depend only on the dimensionality of the system. Aside from a few special cases,(6,7) t is also universal. In two dimensions, which is the focus of our paper, we have $\nu = 4/3$, $t \simeq 1.3$, and(8) $D_B \simeq 1.65$.

Vector transport properties of percolation networks, e.g., their elastic moduli, are more difficult to define and calculate. The main reason is that vector transport properties depend sensitively on the microscopic force laws between the bonds and/or sites of the network and, in principle, one can define a large number of different microscopic force laws between the bonds and/or sites. We consider here a percolation network whose bonds represent elastic elements (springs) that can be stretched and/or bent. The elastic energy of the system is given by(9)

$$E = \frac{\alpha}{2} \sum_{\langle ij \rangle} [(\mathbf{u}_i - \mathbf{u}_j) \cdot \boldsymbol{R}_{ij}]^2 e_{ij} + \frac{\beta}{2} \sum_{\langle jik \rangle} (\delta\theta_{jik})^2 e_{ij} e_{ik} \tag{3}$$

where the first term of the right-hand side represents the contribution of the stretching or central forces (CFs), whereas the second term represents the contribution of the angle-changing or bond-bending (BB) forces. Here, α and β are the central and BB force constants, respectively, $\mathbf{u}_i$ is the (infinitesimal) displacement of site i, $\mathbf{R}_{ij}$ is a unit vector from i to j, and $\langle jik \rangle$ indicates that the sum is over all triplets in which the bonds j–i and i–k form an angle whose vertex is at i. The elastic moduli G of the network can be defined by assigning a finite elastic constant e_{ij} to the present bonds and a zero elastic constant to the absent bonds. If both α and β are nonzero, we have the so-called BB model.$^{(9,10)}$ The percolation threshold of the BB model would be the same as that of scalar percolation defined above if each site of the network interacts with at least $d(d-1)/2$ of its nearest-neighbor sites in d dimensions, which, in practice, is the case. For this model, the elastic moduli G vanish as p_c is approached as

$$G \sim (p-p_c)^{f_{bb}} \tag{4}$$

The exponent f_{bb} is also largely universal; in two dimensions we have$^{(11)}$ $f_{bb} \simeq 3.96$. Moreover, it has been suggested that$^{(12,13)}$ $f_{bb}=t+2\nu$ for any $d \leqslant 6$, in excellent agreement with the numerical estimates.

If $\beta=0$, we have the CF or *rigidity percolation* model in which only stretching or CFs are present. The percolation threshold p_{ce} of this system is *not* the same as that of scalar percolation$^{(14,15)}$ because not every deformation of the network costs an elastic energy E. For example, for a d-dimensional hypercubic network one has $p^B_{ce}=p^S_{ce}=1$, where p^B_{ce} and p^S_{ce} are the bond and site percolation thresholds, respectively. In fact, only if the coordination number of a d-dimensional network is larger than $2d$ does the network have nonzero G for *any* $p>p_{ce}$. For the triangular network we have $p^B_c=2\sin(\pi/18)\simeq 0.347$ and $p^S_c=1/2$, whereas$^{(16,17)}$ $p^B_{ce}\simeq 0.641$ and$^{(18)}$ $p^S_{ce}\simeq 0.713$. Near p_{ce}, the elastic moduli G vanish as

$$G \sim (p-p_{ce})^{f_c} \tag{5}$$

Moreover, a corresponding correlation length ξ_c can also be defined such that near p_{ce}

$$\xi_c \sim (p-p_{ce})^{-\nu_e} \tag{6}$$

and the backbone of the elastic percolation cluster is a fractal object for any length scale less than ξ_c with a fractal dimension D_{EB}.

The universality class of CF percolation and the precise values of ν_c, D_{EB}, and f_c have been controversial for several years. Earlier simulations$^{(19-21)}$ had indicated that for BP in two dimensions, $D_{EB}\simeq 1.95$, $\nu_c\simeq 1.1$, and $f_c\simeq 1.45$. Moreover, it was suggested$^{(21)}$ that SP and BP may not even belong to the same universality class, and may be characterized by their own sets of critical components. For example, we have found that$^{(18,21)}$ $f_c\simeq 1.12$ for SP on the triangular network. More recently, very accurate simulations$^{(16-18)}$ indicated that the critical properties of CF percolation are highly sensitive to the precise value of p_{ce}, and that even for large networks near p_{ce} there are large correction-to-scaling terms such that, e.g., Eq. (5) should be rewritten as

$$G \sim (p-p_{ce})^{f_c}\,[a_1+a_2(p-p_{ce})^{-\Delta_1}+a_2(p-p_{ce})^{-\Delta_2}+\cdots] \tag{7}$$

where the a's are constant, and Δ_1 and Δ_2 are correction-to-scaling exponents. Accordingly, it was found that(16,17) for BP, $f_c(d=2) \simeq 3.9$, which is compatible with that of the BB model, and(16) $D_{EB} \simeq 1.62$, in agreement with that of BB and scalar models. However, these more accurate simulations also yielded(18) $f_c(d=2) \simeq 1.12$ for SP, and a small but significant difference between the values of ν_c for SP and BP was also observed, so that the controversy is still not resolved.

In this paper we consider a percolation problem on CF networks in which both sites and bonds are randomly occupied with probabilities s and b, respectively. This is the so-called site-bond percolation, which, in the case of scalar percolation, has been considered by several authors.(22–24) Here we consider its analog in the CF percolation problem. We develop a three-parameter position-space renormalization group (PSRG) transformation and obtain the global flow diagram for the system. From the flow diagram and the number of nontrivial fixed points of the PSRG transformation (i.e., those that are not zero or unity), we can determine whether SP and BP on CF networks belong to the same universality class. Moreover, we also determine f_c for both SP and BP to see whether they are significantly different.

2. POSITION-SPACE RENORMALIZATION FOR RIGIDITY PERCOLATION

Our PSRG transformation for the site-bond problem is in the spirit of that for the scalar percolation case.(24,25) However, there is a significant difference between our PSRG transformation for site-bond percolation in CF networks and that of scalar percolation,(24,25) and also the PSRG method for the BB model.(10,26) As mentioned above, in the CF percolation problem not every sample-spanning cluster of bonds is rigid and gives rise to nonzero G, because one can deform many different configurations of the network *without* changing the elastic energy E. Thus, the usual rule of RG transformations that *every* sample-spanning cluster of present bonds in the RG cell must be included in the PSRG transformation *cannot* be used. Instead, we adopt the following rule: *only* those sample-spanning clusters of the RG cell that are rigid and have nonzero values of G are included in the PSRG transformation. To find such clusters, we have to calculate the elastic moduli of *all* configurations of the RG cell. This is done by minimizing E with respect to $\mathbf{u}_i$ and solving the resulting set of linear equations that govern the displacements of the internal nodes of the RG cell. Having calculated $\mathbf{u}_i$, we can determine the elastic moduli G. In this sense, our PSRG approach to the CF percolation problem is novel. Moreover, our approach also allows us to calculate f_c, since we calculate the elastic moduli of the RG cell.

Using this rule, we now derive the PSRG transformations for site and bond occupation probabilities. We use a two-cell approach in which a cluster consists of two adjacent cells. This is necessitated by the fact that the spanning clusters have to be rigid and, as pointed out by Shapiro,(25) who studied the scalar site-bond percolation, this sort of approximation yields more accurate results than the one-cell approach of Nakanishi and Reynolds.(24) Thus, to derive the RG transformation for s', the renormalized site occupation probability, the cell shown in Fig. 1a is mapped onto the configuration with one site and three bonds (which is the *only* configuration that can be rigid). Therefore,

$$\begin{aligned} s'b'^3 &= s^3[b^9 + 9b^8 + 30b^7(1-b)^2 + 33b^6(1-b)^3 + 12b^5(1-b)^4] \\ &\equiv R_1(b, s) \end{aligned} \tag{8}$$

Note that *all* sites in the original cell have to be occupied in order for the cell to be rigid. A similar approach is taken for calculating the renormalized bond occupation probability b'. Thus, various configurations of the cell

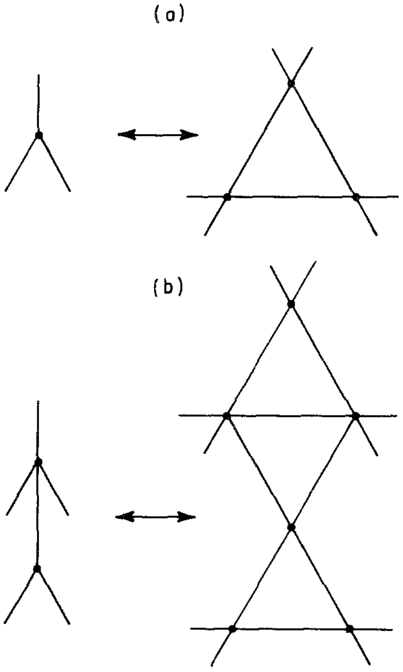

Fig. 1. RG cells used in this paper. (a) One-cell configurations, (b) two-cell configurations.

shown in Fig. 1b, with 6 sites and 18 bonds, are mapped onto one renormalized site. Noting that there are a few configurations of the renormalized cells that can be rigid, we obtain

$$\begin{aligned} &s'^2 b'[b'^5+5b'^4+2b'^3(1-b')^2] \\ &\quad = s^6(b^{18}+18b^{17}+\cdots)+s^5(1-s)(b^{14}+\cdots) \\ &\qquad + s^5(1-s)(b^{13}+\cdots)+s^4(1-s)^2\,(b^{10}+\cdots) \\ &\qquad + \cdots \equiv R_2(b,s) \end{aligned} \tag{9}$$

The rest of the computation is done in the usual way. Equations (8) and (9) provide a complete set of RG transformations for our problem. The fixed points of R_1 and R_2, i.e., the solutions (b^*, s^*) of the equations $sb^3 = R_1(b, S)$ and $s^2b[b^5+5b^4+2b^3(1-b)^2] = R_2(b, s)$, are $(0, 0)$, $(1, 1)$, and $(0.6720, 0.7974)$. The first two sets of fixed points are trivial, while the last one represents an estimate of the true (p_{ce}^B, p_{ce}^S), which are $(0.641, 0.713)$. Thus, our estimate of p_{ce}^B differs from its true value by only 4.8 %, while the difference between our estimate of p_{ce}^S and its true value is about 12 %. If we linearize R_1 and R_2, we obtain

$$\delta b' = 0.5734\delta s + 0.9178\delta b \tag{10}$$

$$\delta s' = 0.9407\delta s + 0.4233\delta b \tag{11}$$

where, e.g., $\delta b = b - b^*$. Equations (10) and (11) have two eigenvalues, $(\lambda_1, \lambda_2) = (1.422, 0.4365)$, but only $\lambda_1 > 1$ is relevant and therefore $v_c = \ln b/\ln \lambda_1$, where $b = 3^{1/2}$ is the scale factor of the cell size (assuming that the length of each bond is unity), which yields

$$v_c \simeq 1.56 \tag{12}$$

The fact that there is only one set of nontrivial fixed points, and only one relevant eigenvalue, means that the *geometrical* exponents of the CF clusters in SP and BP belong to the same universality class. This can be easily seen in Fig. 2, where we show the global flow diagram of the RG transformations. It is seen that the point $(0.672, 0.7974)$ can be reached from both the line $b = 1$ and the line $s = 1$, which is indicative of universality. From the flow diagram we also obtain $b^*(s = 1) = 0.498$ and $s^*(b = 1) = 0.741$. Thus, in this way our estimate of p_{ce}^S improves, while that of p_{ce}^B becomes less accurate. Moreover, in the two-cell approximation and in the limit $s = 1$, we obtain $b^* = 0.41$, while in the limit $b = 1$ we obtain $s^* = 0.87$. The corresponding values of v_c are 1.65 and 1.50, respectively, not much different from one another or from $v_c \simeq 1.56$ found at

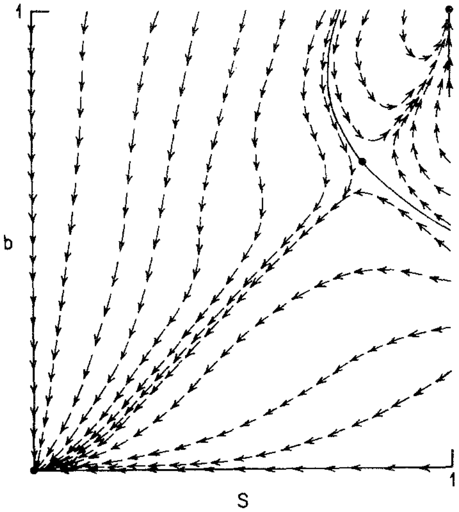

Fig. 2. Global flow diagram for the RG transformations.

(0.672, 0.7974). This is again indicative of the universality of ν_c. Although some of these estimates are not very accurate, the qualitative features of the global flow diagram are usually independent of the values of (b^*, s^*). Since it has already been argued that[16,17] the geometrical exponents of CF clusters in BP on the triangular network belong to the universality class of scalar percolation, the conclusion is that the geometrical exponents of both rigidity BP and SP may belong to the universality class of scalar percolation.

However, the global flow diagram cannot provide any information about the universality of f_c in BP and SP. We can calculate f_c/ν_c using our RG transformation. Following Feng and Sahimi,[10] we calculate an RG transformation for α', the renormalized stretching force constant (or E', the renormalized elastic energy associated with the deformation of the renormalized RG cell). To do this, we impose a fixed displacement on the exterior nodes of the RG cell and calculate the displacements of the internal nodes of the cell, which is what we did in order to derive Eqs. (8) and (9). Using a procedure that was first developed by Bernasconi[27] for percolation conductivity, the RG transformation for α' is approximated by

$$\begin{aligned} &s'^2 b'^6 \ln(2.727\alpha') + s'^2 b'^5 (1-b') \ln(0.8\alpha') + \cdots \\ &\quad = s^6 [b^{18} \ln(2.727\alpha) + b^{17} \ln(2\alpha) + \cdots] + s^5 (1-s)(b^{14} \cdots) + \cdots \qquad (13) \end{aligned}$$

where terms such as $2.727\alpha'$, $0.8\alpha'$,... represent the equivalent stretching force constant of the corresponding configuration of the renormalized RG cell [see Eqs. (8) and (9)]. In the percolation conductivity problem, such terms are replaced by the corresponding conductivity of the RG cell. The critical exponent f_c is then given by(10)

$$\frac{f_c}{v_c}=\frac{\ln(1/\lambda_\alpha)}{\ln b} \tag{14}$$

where $\lambda_\alpha = \partial\alpha'/\partial\alpha$, evaluated at (b^*, s^*, α^*), where α^* is the fixed point of Eq. (13).

Using this approximation, we obtain $f_c/v_c \simeq 2.60$, which, together with $v_c \simeq 1.56$, yields $f_c \simeq 4.05$, which is only 2.4% larger than $f_{bb} \simeq 3.96$ for the BB model. However, even this value of f_c is misleading, because it says nothing about the universality of f_c. A better way of investigating this is to calculate f_c/v_c at (0.41, 1) and (1, 0.87). We find that at (0.41, 1), which corresponds to the pure BP, $f_c/v_c \simeq 3.66$, while at (1, 0.87), the pure SP, $f_c/v_c \simeq 0.91$. These two values differ by a factor of 4, and the difference is of the same order of magnitude as that found in the simulations.(16–18) Moreover, the value of f_c/v_c at (0.41, 1) is close to the numerical estimate(16–18) found for BP, while f_c/v_c at (1, 0.87) is close to the numerical estimate for SP.(18) Since these two values of f_c/v_c are very different, whereas the values of v_c calculated at the same points are not much different (they differ by about 10%), this may be interpreted as a strong indication that f_c is *not* universal, although one should perhaps consider larger RG cells in order to check further this conclusion.

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